BALANCED AND NEARLY BALANCED n-ARY DESIGNS WITH VARYING BLOCK SIZES AND REPLICATIONS

By

A. K. Nigam, S. K. Mehta and S. K. Agarwal
Institute of Agricultural Research Statistics, New Delhi

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Introduction

Balanced n-ary designs were introduced by Tocher [7]. These designs are proper (with constant block sizes) and generally equi-replicate. Practical considerations often dictate the use of designs in varying replications and unequal block sizes (for instance, see Calinski [1] and Pearce [5]). It was shown by Rao [6] that even the non-proper designs with unequal replicates can be balanced in the sense that Var. $(t_i - t_{i'})$ is same for all pairs (i, i'), $i \neq i'$. A necessary and sufficient condition for a design to be balanced is that the $\underset{\sim}{C}$ matrix of the adjusted intrablock normal equations shall have its diagonal elements equal and all its off-diagonal elements equal. If r_i denotes the number of times the ith treatment is replicated, k_j denotes the number of plots in the j^{th} block and $\underset{\sim}{N} = (n_{ij})$ is the incidence matrix, the elements of $\underset{\sim}{C}$ are

$$c_{ii} = r_i - \sum (n^2_{ij}/k_j), \quad i = 1, 2, \ldots, v \qquad \ldots(1.1)$$
$$j = 1, 2, \ldots, b$$

$$c_{ii'} = - \sum (n_{ij}\, n_{i'j}/k_j), \quad i \neq i' \qquad \ldots(1.2)$$

Since the C-matrix of any block design satisfies $\underset{\sim}{C}\,\underset{\sim}{1} = \underset{\sim}{0}$, where $\underset{\sim}{1} = (1, 1, \ldots 1)'$, the equality of off-diagonal elements of $\underset{\sim}{C}$ implies the equality of its diagonal elements.

Kulshreshtha *et al.* [3] presented a method of construction of balanced binary and ternary designs with two block sizes and varying replications (See also, John [2]).

In the present paper the work of Kulshreshtha *et al.* [3] has been generalised to construct balanced n-ary block designs (for any n) with more than two block sizes and unequal replications by using $(n-1)$ *BIB* designs with same number of treatments. This is presented in Section 2. These n-ary designs, however, involve too many replications and as such may not be much useful. To overcome this limitation, in Section 3, 'nearly balanced' designs have been suggested which, while requiring much less number of replications, are seen to be almost as efficient as the totally balanced designs.

2. Balanced n-ary Designs

Consider $(n-1)$ *BIB* designs in same v treatments $t_1, t_2, \ldots, t_v$ and with parameters $(v, b_i, r_i, k_i, \lambda_i)$, $i=1,\ldots, n-1$. Augment each of the b_1 blocks with k'_1 plots, each of the b_2 blocks with k'_2 plots,..., each of the b_{n-2} blocks with k'_{n-2} plots and let all the augmented plots contain the treatment t_0. Further, let all the augmented blocks be repeated m times and the $(n-1)$th *BIB* design be repeated p times. We prove the following.

THEOREM 2.1

The design in $(v+1)$ treatments and

$$\left(m \sum_{\theta=1}^{n-1} b_\theta + pb_{n-1}\right)$$

blocks is a balanced design with

$$r_0^* = m \sum_{\theta=1}^{n-2} b_\theta k'_\theta,\ r_i^* = \left(m \sum_{\theta=1}^{n-2} r_\theta + pr_{n-1}\right),$$

$$i=1,2,\ldots,v,\ k_1^* = k_1 + k'_1,\ k_2^* = k_2 + k'_2 \ldots,$$

$k^*_{n-2} = k_{n-2} + k'_{n-2}$, $k^*_{n-1} = k_{n-1}$, whenever m and p are such that

$$p/m = \left((k_{n-1}/\lambda_{n-1}\right)\left[\sum_{\theta=1}^{n-2} \left\{\left(k'_\theta\, r_\theta - \lambda_\theta\right) / \left(k_\theta + k'_\theta\right)\right\}\right].$$

Proof: The parameters r^*_0, r^*_i and k^*_j need no proof.

Substituting these values in (1. 2), one obtains

$$-m \sum_{\theta=1}^{n-2} \left\{\left(k'_\theta\, r_\theta\right) / \left(k_\theta + k'_\theta\right)\right\} = -m \sum_{\theta=1}^{n-2} \left\{\lambda_\theta / \left(k_\theta + k'_\theta\right)\right\} + p\,\lambda_{n-1}/k_{n-1} \quad \ldots(2.1)$$

Simplifying (2.1), we get the value of p/m as

$$p/m=\left(k_{n-1}/\lambda_{n-1}\right)\left[\sum_{\theta=1}^{n-2}\left\{\left(k_{\theta}^{'}\ r_{\theta}^{'}-\lambda_{\theta}\right)\Big/\left(k_{\theta}+k_{\theta}^{'}\right)\right\}\right] \quad ...(2.2)$$

Hence the theorem.

Remark 1 : The design will be n-ary if max. $(k_j^{'})=n-1$; $j=1, 2,..., n-2$, and $k_{j-1}^{'}=k_j^{'}-1$. Further, trivially, when all $k_j^{'}$ are equal to 2, we get a ternary design and if all $k_j^{'}$ are equal to 1, we get a binary design.

Remark 2 : It may be observed that the $(n-1)$ *BIB* designs with same number of treatments need not all be distinct. We now give an example to illustrate the procedure.

Example 1. Let us obtain a 4-ary balanced design by taking the *BIB* design $v=b=4$, $r=k=3$, $\lambda=2$. For constructing 4-ary design we need a set of 3 *BIB* designs in the same v treatments. Here we shall consider the same *BIB* design three times, so that $v_1=v_2=v_3=4$, $b_1=b_2=b_3=4$, $r_1=r_2=r_3=3$, $k_1=k_2=k_3=3$, $\lambda_1=\lambda_2=\lambda_3=2$.

Now, augment each block of the first *BIB* design with two plots and each block of the second *BIB* design with three plots having treatment t_0, then $k_2^{'}=3$. Substituting the values of various parameters in (2.4), we have

$$p/m=59/20.$$

Thus, by repeating the augmented blocks in the above design 20 times and the unaugmented blocks 59 times, one gets a 4-ary balanced block design in 5 treatments with the parameters $v^*=5$, $b^*=396$, $r_0^*=400$, $r_i^*=297$, $i=1, 2, 3, 4$, $k_1^*=5$, $k_2^*=6$, $k_3^*=3$.

$n_1=80$, $n_2=80$, $n_3=236$, n_1, n_2, n_3 being the number of times the blocks with sizes k_1^*, k_2^*, k_3^* respectively occur in the design.

3. Nearly Balanced Designs

In the example of Section 2, it is observed that since the values of p and m are too large, 396 blocks are required to achieve balance.

The total number of plots required for the adoption of such a design is 1588 which is enormously large. However, if 3/1 is taken as an approximation to 59/20 *i.e.*, p/m is reduced to the nearest integral value, it may still be possible to obtain a design which may be nearly balanced and more realistic. It is thus clear that in the absence of a suitable balanced design it is worthwhile to make a search for a nearly balanced design.

To explain this point further, we work out the variances of all possible treatment comparisons for both the cases *viz.*, (*i*) $p/m=59/20$ and (*ii*) $p/m=3/1$.

Case (*i*)	*Case* (*ii*)
$V_1=\text{Var}\ (t_i-t_i')=0.0073\ \sigma^2$	$\text{Var}\ (t_i-t_i')=0.1466\ \sigma^2$
$i\neq i'$; $i, i'=0, 1, 2, 3, 4$	$i\neq i'$; $i, i'=1, 2, 3, 4$
	$\text{Var.}\ (t_o-t_i)=0.1475\ \sigma^2, i=1, 2, 3, 4$
	$V_2=$Average variance
	$=[(6\times 0.1466+4\times 0.1475)/10]\ \sigma^2$
	$=0.1470\ \sigma^2$.

It can be seen that there is little variation in the variances of different treatment comparisons in case (*ii*) as these range from $0.1466\ \sigma^2$ to $0.1475\ \sigma^2$ only.

While making objective comparison between the cases (*i*) and (*ii*), one notices that the design in case (*i*) is based on $N_1=1588$ plots whereas in case (*ii*), it is based on $N_2=80$ plots. We must, therefore, compare N_1V_1 and N_2V_2. Now $N_1V_1=11.5924\ \sigma^2$ and $N_2V_2=11.7600\ \sigma^2$; and, these values are quite comparable. Therefore, the design with $p/m=3$ can be taken as good as a balanced design and without any significant loss in the precision.

One can easily work out more examples of nearly balanced designs. A useful catalogue of binary nearly balanced designs has been given by Nigam [4].

SUMMARY

In this note a method of construction of balanced n-ary designs with varying block sizes and varying replications has been presented. These designs require too many experimental units. To reduce the size of the experiment, nearly balanced designs have been suggested.

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